# A continuum model for granular materials: Considering dilatancy and the Mohr-Coulomb criterion

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**Summary.** In this paper we will explore the consequences of the Mohr-Coulomb criterion on the constitutive equation proposed by Rajagopal and Massoudi [1]. This contunuum model which is based on the earlier works of Cowin [2] has also the ability to predict the dilatancy effect which is related to the normal stress effects. At the same time, if a proper representation is given to some of the material parameters, this model would also comply with the Mohr-Coulomb criterion. We also present, as a special case, an exact solution for the case of simple shear flows.

## 1 Introduction

Perhaps the earliest study of granular flow is the hourglass or sand clock, which was in common use in the early 14th century for measuring the speed of ships. These devices were used, during the Middle Ages, by scholars to regulate their studies and by the clergy to time their sermons [3]. Another old example is the art of ploughing which has been used throughout centuries. The knowledge of our ancestors was more practical than theoretical; thus, with industrialization, the animals were replaced by tractors, and engineers who had designed these tractors were surprised to find that the drag of a plough is almost independent of speed [4]. That is, to everyone's surprise, it was found that ploughing at greater speeds does not require greater forces. Phenomena such as debris flow [5] which is the agent for forming alluvial cones in the mounts of mountain canyons [6], [7] and snow or ice avalanches [8], [9] are among the most threatening natural phenomena in some regions of the world. Other examples of bulk solids are coal, sand, ore, mineral concentrate, crushed oil shale, grains, cereals, animal feed, and powders.

Early studies of the flow of bulk solids were mainly concerned with the engineering and structural design of bins and silos. The inaccuracies of these theories, especially for dynamic conditions of loading or emptying, occasionally resulted in the failure of the bin or silo [10], [11]. Furthermore, flowing granular materials is the limiting case of two-phase flow at high solid concentration and high solid-to-fluid density ratios. Many situations, such as discharge through bin outlets [12], flow through hoppers and chutes, flow of mixtures, and slurry transport, require information on flow patterns, strength of powders, and their adherence to surfaces [13]. It is very difficult to characterize bulk solids, which are composed of a variety of materials. This is mainly due to the fact that small variations in some of the primary properties of the bulk solids such as the size, shape, hardness, particle density, and surface roughness, can result in very different behavior. Furthermore, secondary factors such as the presence or absence of moisture, the severity of prior compaction, the ambient temperature, etc., which are not associated directly with the particles, can have significant effects on the behavior of the bulk solids.

Although the fluid phase plays an importent role in determining the dynamics of dilute suspensions, it does not have much influence on bulk solids behavior. That is, when the solid phase is dominant, the behavior of the bulk materials, in general, is governed by interparticle cohesion, friction, and collisions. In some cases the effects of the interactions between the fluid and solid constituents may be small because the interstitial fluid has relatively small density and viscosity (e.g., a gas). When the effects or the presence of the fluid phase cannot be ignored, then one has to resort to multiphase or multicomponent modeling, by considering the interaction mechanisms between the two phases.

A powder is composed of particles up to 100 $\mu$m (diameter) with further sub-division into ultra fine (0.1 to 1 $\mu$m), superfine (1 to 10 $\mu$m), or granular (10 to 100 $\mu m$) particles. A granular solid consitsts of materials ranging from about 100 to 3000 $\mu$m [13]. A granular material covers the combined range of granular powders and granular solids with components ranging in size from about 10 $\mu$m up to 3 mm. This range includes most of the materials used in laboratory experiments and whenever we use the term granular material we shall henceforth refer to this range. Brown and Richards [13] define a bulk solid as: *"An assembly of discrete solid components dispersed in a fluid such that the constituents are substantially in contact with near neighbors. This definition excludes suspensions, fluidized beds, and materials embedded in a solid mixture."*

The basic and fundamental question in modeling the granular materials is whether a single constitutive relation for the Cauchy stress tensor $\boldsymbol{T}$ is sufficient to describe the various flow regimes and geometries. In addition to this, whether one decides to use classical continuum theories or a modified version of kinetic theory of gases as applied to solid macroscopic materials, or computer simulation based on particle dynamics [14], or experimental observations leading to phenomenological relations for the stress tensor [15], etc., has added to the complexity and diversity of this field of research, by producing many different forms for the stress tensor. At the present time, there is no unified theory for granular materials.

Any theory attempting to describe the behavior of flowing granular materials should embody several features, some of which are unique to granular materials. For example, a bulk solid is not exactly a solid continuum since it takes the shape of the vessel containing it; it cannot be considered a liquid for it can be piled into heaps; and it is not a gas for it will not expand to fill the vessel containing it. Perhaps the phase that the bulk solids most resemble is that of a non-Newtonian fluid. Therefore, it seems reasonable to expect a theory for flowing granular materials to exhibit characteristics unique to viscoelastic fluids such as the normal stress effects.

From a continuum mechanics point of view, there are many different approaches that one can take. From the observation/experimental point of view, the pioneering work of Bagnold [16], [17] has led to many formulations of non-Newtonian models [18], [19], [20], [21], [22], [23]. For a review of this aspect of the modeling activities we refer the reader to the recent article by Elaskar and Godoy [24]. The fact that granules can flow has prompted many investigators to look at the flow of particles as a fluid phenomenon, even as a compressible fluid [25], [26], [27]. At the same time, there have been many attempts to formulate or to propose rate-independent theories [28], plasticity theories [29], viscoelastic theories [30], hypoplastic theories [31]. Theories with microstructure have also been proposed [32], [33], [34], [35]. There have also been attemps to include the effect of "fluctuation" of the particles into the stress tensor formulation [36], [37]. At the same time, others have shown the similarity between the rapid flows of granular materials and the turbulent motion of a fluid [38], [39]. A general continuum theory with thermodynamical restrictions was proposed by Goodman and Cowin [40], [41]. This work has subsequently been modified, extended, and generalized by various researchers [42], [43], [44].

The outline of this paper is as follows: we first review, very briefly, some basic concepts such as yield criterion and dilatancy in granular materials. These are primarily non-linear effects which might be present in non-linear solids and/or non-linear fluids. Next we will look at a continuum model first proposed by Cowin [45], and later modified by Savage [20] (and many others). The specific version that is studied in this paper is the one proposed by Rajagopal and Massoudi [1]. In the last section, we will look at the effects of imposing or expecting this model to comply with Mohr-Coulomb criterion and dilatancy. There are many excellent review articles where many of the important issues relevant to granular materials are discussed. These recent articles take a general perspective and present a review of statistical theories (kinetic theory of gases, computer simulation) and continuum theories. We refer the reader to the articles by Nedderman et al. [12], Savage [3], Hutter and Rajagopal [46], Jaeger et al. [47], de Gennes [48], Hermann and Luding [14], and the book by Mehta [49].

### *1.1 Cohesionless and cohesive materials*

The mechanical properties of materials such as soils range between those of plastic clay [50] and those of clean, perfectly dry sand. Slopes of all kinds, including river banks and sea coast bluffs, hill, mountains, etc., remain in place because of the shearing strength possessed by the soil or rock. If we dig into a bed of dry (or completely immersed) sand, the material at the sides of the hole would slide toward the bottom. This behavior indicates a complete absence of a bond between the individual particles [51]. This sliding continues until the angle of inclination of the slopes becomes equal to a certain angle known as the "angle of repose." Brown and Richards [13] define two angles of repose as:

*"The angle to the horizontal assumed by – the free surface of a heap at rest and obtained under stated conditions:*

*(i) the poured angle of repose is formed by pouring the bulk solid to form a heap below the pour point.*

*(ii) The drained angle of repose is formed by allowing a heap to emerge as superincumbent powder is allowed to drain away past the periphery of a horizontal flat platform previously buried in the powder."*

Various techniques to measure the angle of repose are given by Weighardt [4]. Very often it is taken for granted that the angle of repose, $\gamma$ is the same as the angle of internal friction, $\phi$. Theory can only say that the slope of the pile of sand cannot be steeper than $\phi$, or $\gamma \geq \phi$. This internal angle of friciton is related to the amount of cohesion present in the material. In simple terms, the bond between the particles, cohesion, is influenced by a variety of forces including Van der Waals' forces, Coulomb forces, and capillary forces [52]. A definite angle of repose cannot be assigned to a granular material with cohesion, since the steepest angle at which such a material can stand decreases with increasing height of the slope [51]. The mechanical properties of real granular materials are so complex that a rigorous mathematical analysis of their behavior seems impossible. Therefore, many branches of applied mechanics, such as theoretical soil mechanics deal exclusively with the behavior of idealized granular materials ranging from ideal sands (cohesionless granular material) to ideal clays (ideally cohesive meterial, i.e., no internal friction) [53], [54].

To have a better understanding of the physical interpretation of the angle of internal friction, we envision a block of solid material resting on an inclinded plane at an angle $\phi$ with the horizontal line. If the block is to slide on the surface, we must have

$$P_f = P_n \tan\phi \tag{1}$$

where $P_n$ is the normal force acting on the block, $P_f$ is the friction force, $\phi$ is the angle of friction and $\tan\phi$ is the coefficient of friction. The characteristic $\phi$ is a property of the materials that are in contact.

In sands and other cohesionless granular materials, a similar relationship exists between the force required to overcome all frictional resistance and cause slip on a plane through a mass of granular material. This relation can be written as [55]:

$$P_s = P_n \tan\phi\,, \tag{2}$$

where $P_n$ is defined as the normal force on the plane subject to slip, $\phi$ is now called the internal angle of friction, and $P_s$ is the shearing force that causes yielding. Of course, the frictional resistance in granular materials is more complex than that between solid bodies, since it is due to both sliding friction and rolling friction.

### *1.2. Mohr-Coulomb criterion*

A criterion often used when devising theory for the flow of granular materials is that the equilibrium states specified by the theory are required (or are shown) to coincide with the limiting equilibrium states specified by the Mohr-Coulomb criterion. The Coulomb failure criterion [56], [57], [58] based on experiments, state that yielding will occur when

$$|S| = bT + c\,, \tag{3}$$

where $S$ and $T$ are the shear stress and normal stress, respectively, acting on a plane at a point; $c$ is a coefficient of cohesion; and $b$ is a coefficient of static friction related to the internal angle of friction $\phi$ through

$$b = \tan\phi\,. \tag{4}$$

When cohesion is absent ($c = 0$), it is usual to call a granular medium an ideal one. One in which internal friction is absent ($\phi = 0$), is called an ideally cohesive medium. For dry, coarse materials, the cohesion coefficient can be neglected. Typical values for the internal angle of friction $\phi$, obtained during quasi-static yielding at low stress levels are close to the angle of repose, e.g. about 24° for spherical glass beads and 38° for angular sand grains [13].

### *1.3 Dilatancy*

A unique property of granular materials was observed by Reynolds [59] who named it "dilatancy". The concept of dilatancy is generally taken to be the expansion of the voidage that occurs in a tightly packed granular arrangement when it is subjected to a deformation. Reynolds [59] used the idea of dilatancy in describing a familiar phenomenon in sand:

*"At one time the sand will be so firm and hard that you may walk with high heels without leaving a footprint; while at others, although the sand is not dry, one sinks in so as to make walking painful. Had you noticed, you would have found that the sand is firm as the tide falls and*

*becomes soft again after it has been left dry for some hours. The tide leaves the sand, though apparently dry on the surface, with all its interstices perfectly full of water which is kept up to the surface of the sand by capillary attraction; at the same time the water is percolating through the sand from the sands above where the capillary action is not sufficient to hold the water. When the foot falls on this water-saturated sand, it tends to change its shape, but it cannot do this without enlarging the interstices – without drawing in more water. This is a work of time, so that the foot is gone again before the sand has yielded."*

Many of the existing theories for flowing granular materials use this observation to relate the applied stress to the voidage and the velocity. One of the first and most interesting observations of the realtionship between the stress in granular materials and voidage was also given by Reynolds [60]:

*"Taking a small indiarubber bottle with a glass neck full of shot and water, so that the water stands well into the neck. If instead of shot the bag were full of water or had anything of the nature of a sponge in it, when the bag was squeezed, the water would be forced up the neck. With the shot the opposite result is obtained; as I squeeze the bag, the water decidedly shrinks in the neck...When we squeeze a sponge between two planes, water is squeezed out; when we squeeze sand, shot, or granular material, water is drawn in."*

The idea of dilatancy of granular materials can be simply explained for an idealized case: in order for a shearing motion to occur in a bed of closely packed spheres, the bed must expand by increasing its void volume (cf. Fig. 1). The work of Reynolds was followed by the experimental studies of Jenkin [61], Rowe [62], Andrade and Fox [63], Reiner [64], and [Bolton [65] to name a few. Many attempts have been made to include the effects of dilatancy in the theory (Nixon and Chandler [66] for a plasticity theory, Mehrabadi et al. [67] from a micromechanical point of view, Goddard and Bashir [68], and Goddard [69] from a rheological perspective). In fact, Reiner [18] was one of the first who used a non-Newtonian model to predict "dilatancy" in wet sand.

## 2 Objectives

Firstly, we present the development of a constitutive relation for the stress tensor due to Rajagopal and Massoudi [1], and then show that with this representation it is possible to observe the dilatational effects. This is due to the nonlinear terms in the constitutive equation which

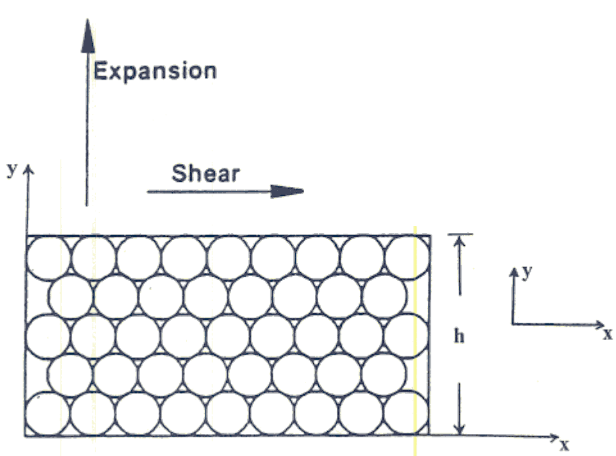


**Fig. 1.** An illustration of dilatancy in an ensemble of initially close-packed spheres

give rise to the normal stress effects. Secondly, we show that if we relate some of the material parameters in this constitutive equation to cohesion and internal angle of friction, this model is also capable of complying with a (limited) form of the Mohr-Coulomb criterion, as pointed out by Cowin [2]. To show the presence of normal stress differences, we solve a simple shear flow problem and present an exact solution for a special case. This model was also used by Rajogopal et al. [44] in their study where they proposed to use an orthogonal rheometer to measure some of these (rheological) parameters.

### *2.1 A constitutiver relation for the stress tensor*

In this section, we give a simple derivation, based on standard techniques of continuum mechanics, for the stress tensor. This is a modified form of an equation which was proposed by Goodman and Cowin [40], [41], subsequently revised and modified by many researchers [cf. Hutter and Rajagopal [46] for a review of theories for granular materials]. Unlike Goodman and Cowin who also tried to use thermodynamical arguments, our presentation will be limited to a purely mechanical case, where the effects of temperature, chemical reactions, and electromagnetic effects are all ignored. We assume

$$\boldsymbol{T} = \boldsymbol{T}(\varrho, \operatorname{grad} \varrho, \boldsymbol{u}, \operatorname{grad} \boldsymbol{u})\,, \tag{5}$$

where $\varrho$ is the bulk density of the material, and $\boldsymbol{u}$ is the velocity. The bulk density $\varrho$ is related to the pure density of the material $\varrho_s$ through

$$\varrho = \varrho_s \nu\,, \tag{6}$$

where $\nu(\boldsymbol{x}, t)$ is a volume distribution funciton (sometimes called volume fraction of solids). The motion of the body and the process of homogenization are shown in Fig. 2. Application of the principle of material frame-indifference (cf. Truesdell and Noll [70]) to Eq. (5) implies

$$\boldsymbol{T} = \boldsymbol{T}(\varrho, \operatorname{grad} \varrho, \boldsymbol{D})\,, \tag{7}$$

where $\boldsymbol{D}$ is the symmetric part of the velocity gradient,

$$\boldsymbol{D} = \frac{1}{2}\left[\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^T\right]. \tag{8}$$

Defining a symmetric tensor, of rank 2, $\boldsymbol{N}$ as

$$\boldsymbol{N} = \operatorname{grad} \varrho \otimes \operatorname{grad} \varrho\,, \tag{9}$$

an isotropic representation of Eq. (7) is (cf. Serrin [71],Truesdell and Noll [70], Spencer [91])

$$\begin{aligned}\boldsymbol{T} = {} & a_0 \mathbf{1} + a_1 \boldsymbol{D} + a_2 \boldsymbol{N} + a_3 \boldsymbol{D}^2 + a_4 \boldsymbol{N}^2 + a_5(\boldsymbol{DN} + \boldsymbol{ND}) \\ & + a_6(\boldsymbol{D}^2\boldsymbol{N} + \boldsymbol{ND}^2) + a_7(\boldsymbol{DN}^2 + \boldsymbol{N}^2\boldsymbol{D}) + a_8(\boldsymbol{D}^2\boldsymbol{N}^2 + \boldsymbol{N}^2\boldsymbol{D}^2)\,,\end{aligned} \tag{10}$$

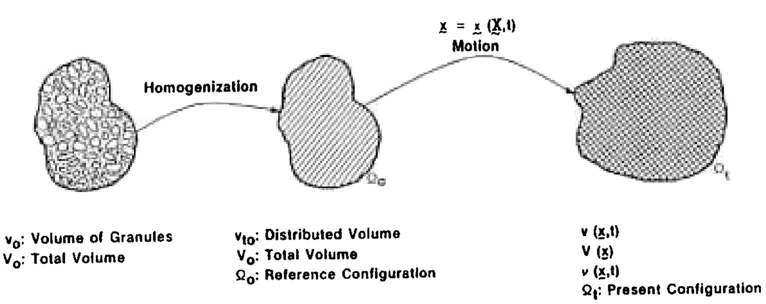


Fig. 2. Motion of the body

where $a_0$ through $a_8$ depend on [91]

$$\varrho, \operatorname{tr} \boldsymbol{D}, \operatorname{tr} \boldsymbol{D}^2, \operatorname{tr} \boldsymbol{D}^3, \operatorname{tr} \boldsymbol{N}, \operatorname{tr} \boldsymbol{N}^2, \operatorname{tr} \boldsymbol{N}^3, \operatorname{tr} (\boldsymbol{D}\boldsymbol{N}),$$
$$\operatorname{tr} (\boldsymbol{D}\boldsymbol{N}^2), \operatorname{tr} (\boldsymbol{N}\boldsymbol{D}^2), \operatorname{tr} (\boldsymbol{D}^2\boldsymbol{N}^2) . \tag{11}$$

Assuming that all the terms that have cross products of $\boldsymbol{D}$ and $\boldsymbol{N}$ and terms of order higher than two in $\boldsymbol{D}$ and $\boldsymbol{N}$ can be neglected, Eq. (10) reduces to

$$\boldsymbol{T} = a_0 \mathbf{1} + a_1 \boldsymbol{D} + a_2 \boldsymbol{N} + a_3 \boldsymbol{D}^2 + a_4 \boldsymbol{N}^2 \tag{12}$$

where $a_0$, $a_1$, ..., etc., are now functions of only the appropriate invariants in keeping with the above assumptions.

Observing that

$$\operatorname{tr} \boldsymbol{N} = \operatorname{grad} \varrho \cdot \operatorname{grad} \varrho = |\operatorname{grad} \varrho|^2$$

$$\boldsymbol{N}^2 = (\operatorname{tr} \boldsymbol{N})\, \boldsymbol{N},$$

Eq. (12) therefore can be re-written as

$$\boldsymbol{T} = a_0 \mathbf{1} + a_1 \boldsymbol{D} + [a_2 + a_4 (\operatorname{tr} \boldsymbol{N})]\, \boldsymbol{N} + a_3 \boldsymbol{D}^2 , \tag{15}$$

where $a_0 - a_4$ depend on the appropriate invariants given by Eq. (11). As it is, Eq. (15) represents a general nonlinear constitutive relation for a material which is flowing and distributing (i.e., re-arranging) itself as it is flowing. If, for simplicity we assume :

$$a_0 = \beta_0'(\varrho) + \beta_1'(\varrho)\, \nabla\varrho . \nabla\varrho + \beta_2'(\varrho) \operatorname{tr} \boldsymbol{D},$$
$$a_1 = \beta_3'(\varrho),$$
$$a_2 + a_4 (\operatorname{tr} \boldsymbol{N}) = \beta_4'(\varrho, \nabla\varrho), \tag{16}$$
$$a_5 = \beta_5'(\varrho),$$

then Eq. (15) can be re-written as:

$$\boldsymbol{T} = [\beta_0'(\varrho) + \beta_1'(\varrho)\, \nabla\varrho \cdot \nabla\varrho + \beta_2'(\varrho) \operatorname{tr} \boldsymbol{D}]\, \mathbf{1} + \beta_3'(\varrho)\, \boldsymbol{D} + \beta_4'(\varrho, \nabla\varrho) + \beta_5'(\varrho)\, \boldsymbol{D}^2$$

A similar expression was proposed by Korteweg to describe the structure of capillarity. The classical theory of capillarity specifies a jump condition at the surface separating homogeneous fluids possessing different densities. Instead, Korteweg (1901) (cf. [70]) proposed a smooth constitutuve relation for the stress that depends on density gradients. This equation is given in Truesdell and Noll [70]:

$$\boldsymbol{T} = -p\mathbf{1} + \lambda (\operatorname{tr} \boldsymbol{D})\, \mathbf{1} + \gamma(\Delta\varrho)\, \mathbf{1} - \alpha(\nabla\varrho \cdot \nabla\varrho)\, \mathbf{1} + 2\mu \boldsymbol{D} - \beta \nabla\varrho \otimes \nabla\varrho + \delta \nabla(\nabla\varrho), \tag{18}$$

where $\Delta$ is the Laplacian operator, $p$, $\lambda$, $\mu$, $\alpha$, $\beta$, $\gamma$, and $\delta$ are functions of $\varrho$ and temperature. Of course, Eq. (18) does not resemble Eq. (17) in that the terms involving $\Delta\varrho$ and $\nabla(\nabla\varrho)$ are higher order terms. In order to get these terms within the context of our derivation, we would have to assume in Eq. (5) that $\boldsymbol{T}$ also depends on grad (grad $\varrho$).

Now, if the grains are incompressible in the sense that their pure density is constant, i. e., $\varrho_s$ = constant, then, we can use Eq. (6) in Eq. (17) and obtain

$$\boldsymbol{T} = [\beta_0(\nu) + \beta_1(\nu)\, \nabla\nu \cdot \nabla\nu + \beta_2(\nu) \operatorname{tr} \boldsymbol{D}]\, \mathbf{1} + \beta_3(\nu)\, \boldsymbol{D} + \beta_4(\nu, \nabla\nu)\, \nabla\nu \otimes \nabla\nu + \beta_5(\nu)\, \boldsymbol{D}^2$$

where the material moduli are now functions of $\nu$. To this equation, Rajagopal and Massoudi (1990) have given the following interpretation: the material moduli $\beta_3(\nu)$ denotes the viscosity (i. e., the resistance of the material to flow): $\beta_1(\nu)$ and $\beta_4(\nu)$ are material parameters that reflect the distribution of the granular solids; and $\beta_0(\nu)$ plays the role akin to pressure in a compressible fluid and is given by an equation of state. The material modulus $\beta_2(\nu)$ is again a viscosity similar to the second coefficient of viscosity in a compressible fluid, and $\beta_5(\nu)$ is similar to what is referred to as "cross-viscosity" in a Reiner-Rivlin fluid [18], [72].

Since there are many material parameters involved, it is essential to devise experiments that will help measure these material moduli. Rajagopal and Massoudi [1] discussed a method for determining these material moduli by using an orthogonal rheometer. The orthogonal rheometer essentially consists of two parallel plates rotating about non-coincident axes with the same angular speed. Because of normal stress differences that develop in the granular materials owing to such a motion, forces and moments are necessary to keep the two plates apart at a constant distance. By measuring these forces and moments, we can characterize the moduli of the material. Such as instrument has been used to characterize the material moduli of viscoelastic fluids.

In the next section, we look at the behavior of a material whose constitutive relation is given by Eq. (19) in a simple shear flow. A discussion for possible dependence at $\beta_0 - \beta_5$ will also be provided.

### 2.2 *Simple shear flow*

Let us consider a simple shear flow. The velocity field $\boldsymbol{u}$ and the volume function $\nu$ are assumed to be of the form (cf. Fig. 1)

$$\boldsymbol{u} = u(y)\,\boldsymbol{i}, \qquad \nu = \nu(y)\,. \tag{20}$$

It then follows that

$$\boldsymbol{D} = \frac{1}{2}\begin{pmatrix} 0 & u' & 0 \\ u' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

$$\boldsymbol{D}^2 = \frac{1}{4}\begin{pmatrix} (u')^2 & 0 & 0 \\ 0 & (u')^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

Also, notice that

$$\nabla\nu\cdot\nabla\nu = \left(\frac{d\nu}{dy}\right)^2 \tag{23}$$

$$\operatorname{tr}\boldsymbol{D} = 0\,, \tag{24}$$

$$\nabla\nu\otimes\nabla\nu = \left(\frac{d\nu}{dy}\right)^2 \boldsymbol{j}\otimes\boldsymbol{j} \tag{25}$$

Now, using Eq. (21)–(25) in (19), we find that

$$T_{xy} = \frac{1}{2}\left[\beta_3(\nu)\,\frac{du}{dy}\right], \tag{26}$$

$$T_{xx} - T_{yy} = -[\beta_4(\nu)]\left(\frac{d\nu}{dy}\right)^2$$

$$T_{yy} - T_{zz} = [\beta_4(\nu)]\left(\frac{d\nu}{dy}\right)^2 + [\beta_5(\nu)]\left(\frac{du}{dy}\right)^2$$

Therefore, we can see from Eq. (27) and (28) that the material exhibits both normal stress differences. If either the term $\beta_5(\nu)\,\boldsymbol{D}^2$ or $\beta_4(\nu)\,\nabla\nu\otimes\nabla\nu$ were absent from the constitutive expression in Eq. (19), the model would be capable of exhibiting only one of the normal stress differences. For example, in an idealized shear flow, it is possible to have constant solid volume fraction. In such a case the term corresponding to $\beta_4(\nu)\,\nabla\nu\otimes\nabla\nu$ vanishes and only one of the normal stress differences, Eq. (28), remains.

As we mentioned in the previous section, the fact that the normal stress differences are nonzero, implies that this model can predict the dilatational effect observed by Reynolds. Having said this, let us see if we can obtain an exact solution for the flow field, given by Eq. (20). We notice that the conservation of mass is automatically satisfied.

The balance of linear momentum is

$$\operatorname{div}\boldsymbol{T} + \varrho\boldsymbol{b} = \varrho\,\frac{d\boldsymbol{u}}{dt}\,, \tag{29}$$

where $\boldsymbol{b}$ is the specific body force field, and $d/dt$ denotes the material time derivative. On substituting Eq. (19) and (21)–(25) into Eq. (29), we obtain

$$\frac{d}{dy}\left[\frac{1}{2}\,\beta_3(\nu)\,\frac{du}{dy}\right] + \varrho b_x = 0\,, \tag{30}$$

$$\frac{d}{dy}\left\{\beta_0(\nu) + [\beta_1(\nu) + \beta_4(\nu)]\left(\frac{d\nu}{dy}\right)^2 + \frac{\beta_5(\nu)}{4}\left(\frac{du}{dy}\right)^2\right\} + \varrho b_y = 0\,, \tag{31}$$

where $b_x$, $b_y$ and $b_z$ are the components of the external body force. Thus, we see that a motion of the form in Eq. (20) is only possible if the $z$-component of the body force field is zero.

$$\varrho b_z = 0$$

We shall now make some assumptions on the material functions $\beta_0(\nu)$, $\beta_1(\nu)$, …etc. We expect that these functions would decrease monotonically with $\nu$. We mentioned earlier that $\beta_0(\nu)$ plays a similar role to that of the pressure in a compressible gas $P(\varrho)$, with $\nu$ now playing the role of density. Assuming a form similar to that for ideal gases leads us to conclude that $\beta_0$ varies linearly with $\nu$ [see Rajagopal and Massoudi [1] for a simple derivation, where they use the idea of densification of the material in the lower regions]. The works of Walton and Braun [73], [74] assume that the viscosity is a function of both the solid fraction $\nu$ and the tensor $\boldsymbol{D}$, and varies as a quadratic function of $\nu$, with $\boldsymbol{D}$ being held fixed. However, the work of Jenkins and Savage [75] based on kinetic theory suggests that $\mu$ is nearly linear in $\nu$.

We shall assume that both $\beta_3(\nu)$ and $\beta_5(\nu)$ are quadratic in $\nu$. Thus, let

$$\beta_3(\nu) = \beta_{30} + \beta_{31}\nu + \beta_{32}\nu^2\,, \tag{33}^{*}$$

* Other forms of this relationship have been proposed, based on experiments, by other investigators. For example, Savage [20] used $\mu = \mu_1[(\nu_m - \nu_0)/(\nu_m - \nu)]^8$ where $\mu_1$ is constant, $\nu_m$ corresponds to the densest possible concentration and $\nu_0$ is the concentration at which fluidity occurs. Passman et al. [76] used $\mu = (\mu_1\nu^2)/(\nu_m - \nu)^2$ where $\mu_1$ is a constant.

$$\beta_5(\nu) = \beta_{50} + \beta_{51}\nu + \beta_{52}\nu^2 , \tag{34}$$

$$\beta_0(\nu) = \varkappa_1 \nu \tag{35}$$

$$\varkappa_1 = \text{constant} , \tag{36}$$

where $\beta_{30}$, $\beta_{31}$, $\beta_{32}$, $\beta_{50}$, $\beta_{51}$, $\beta_{52}$, and $\varkappa_1$ are constants.

We do not have any rationale to assume the form of $\beta_1(\nu)$, $\beta_2(\nu)$, and $\beta_4(\nu)$. Once again, assuming they are smooth functions of $\nu$, and using a Taylor's series expansion and neglecting terms of order $\nu^3$ and higher, we find

$$\beta_1(\nu) = \beta_{10} + \beta_{11}\nu + \beta_{12}\nu^2 , \tag{37}$$

$$\beta_2(\nu) = \beta_{20} + \beta_{21}\nu + \beta_{22}\nu^2 , \tag{38}$$

$$\beta_4(\nu) = \beta_{40} + \beta_{41}\nu + \beta_{42}\nu^2 , \tag{39}$$

where $\beta_{10}$, $\beta_{11}$, $\beta_{12}$, $\beta_{20}$, $\beta_{21}$, $\beta_{22}$, $\beta_{40}$, $\beta_{41}$, and $\beta_{42}$ are constants.

Further restrictions on the coefficients $\beta_3(\nu)$, $\beta_5(\nu)$, and $\beta_2(\nu)$ can be found using the following argument (see Rajagopal and Massoudi [1] for details). If there are no particles, i.e., if $\nu = 0$, the stress tensor $\boldsymbol{T}$ should vanish. Therefore the dependence of $\beta_3$, $\beta_5$, and $\beta_2$ on $\nu$ should be such that as $\nu \to 0$, these functions also vanish (We refer to this as the limiting principle).

Therefore, we must have

$$\beta_{30} = \beta_{50} = \beta_{20} = 0 . \tag{40}$$

Equations (30) and (31) represent a system of two coupled second order nonlinear ordinary differential equations. An appropriate set of boundary conditions for the problem is

$$u(0) = 0 , \quad u(h) = V , \tag{41}$$

$$\nu(0) = \nu_0 , \quad \nu(h) = \nu_h . \tag{42}$$

The values for $\nu(0)$ and $\nu(h)$ can be specified if the data for the volume distribution function are available. [Another possibility is specifying the mass flow rate, if this is known; however this is an integral condition. If we are interested in the flow of granular materials down an incline plane due to gravity for example, we would specify that the material is stress free at $y = h$, and we would specify the mass flow rate.] For the lower plate, we have assumed the no-slip condition, and for the upper plate, the velocity is the same as the shearing velocity. Boundary condition (42) can also be envisioned as having glued particles to the plates, and thus imposing a distribution for $\nu$. In general, therefore, we have to solve this boundary value problem numerically. However, in the absence of body forces, and the special case corresponding to:

$$\beta_3(\nu) = \beta_{31}\nu , \quad \beta_5(\nu) = \beta_{51}\nu , \quad \beta_{11} = \beta_{12} = \beta_{41} = \beta_{42} = 0 . \tag{43}$$

Equations (30) and (31) can be integrated to give

$$\frac{1}{2}\beta_{31}\nu u' = C_1 , \tag{44}$$

$$\varkappa_1\nu + (\beta_{10} + \beta_{40})(\nu')^2 + \frac{\beta_{51}}{2\beta_{31}} \cdot \frac{C_1^{\,2}}{\nu} = C_2 , \tag{45}$$

where $C_1$ and $C_2$ are constants. Equations (44) and (45) admit the solution

$$\alpha y$$

$$\alpha \quad \text{constant}$$

$$\text{constant}$$

However, the system of Eq. (44) and (45) is nonlinear and might admit additional solutions. Also, in general, we would like to assume that the material moduli have the more general representation of Eqs. (33)–(39) subject to the constraint (40). In this case, Eq. (30) and (31) would have to be solved numerically. Once $u$ and $\nu$ are determined, we can use Eq. (26), (27), and (28) to find the normal stress differences, and the shear stress function. If these can be measured experimentally, we could use them to determine the material properties $\beta_3$, $\beta_4$, and $\beta_5$. Of course, the above problem is only an illustration in which we have assumed a very specific structure for the material functions and have ignored the body force field. However, in principle, we could use a similar procedure for other problems.

### 2.3 *The Mohr-Coulomb criterion*

The interpretation that we have given to the material parameters $\beta_0 - \beta_5$ in Eq. (19) in the previous section is a rheological one. However, if the material has not started to flow, the question of yielding and the onset of the flow remains. To look at this issue, as suggested by others [2], [20], [77], [78] we decompose the stress tensor into an equilibrium (or static) part and a dynamic part. Thus, we re-write Eq. (19) as

$$\boldsymbol{T} \quad \boldsymbol{T}_e \quad \boldsymbol{T}_d \tag{48}$$

where

$$\boldsymbol{T}_e \quad [\beta_0(\nu) + \beta_1(\nu)\,\nabla\nu\cdot\nabla\nu]\,\mathbf{1} + \beta_4\nabla\nu\otimes\nabla\nu$$

$$\boldsymbol{T}_d \quad [\beta_2(\nu)\ \ tr\boldsymbol{D}]\,\mathbf{1} + \beta_3(\nu)\,\boldsymbol{D} + \beta_5(\nu)\,\boldsymbol{D}^2\,.$$

It should be remarked that we have not given a rigorous definition of the 'equilibrium' state We understand the term "equilibrium" in the sense that as $\boldsymbol{D}\to\mathbf{0}$, $\boldsymbol{T}\to\boldsymbol{T}_e$.
Cowin [2], [45] and Savage [20] showed that the stress $\boldsymbol{T}_e$, specified by the following equation:

$$\boldsymbol{T}_e \quad (c\cot\phi \quad q)\,\mathbf{1} \quad 2\alpha\boldsymbol{M}$$

satisfies the Coulomb failure criterion. In Eq. (51)

$$q \quad \alpha\left(\frac{}{\sin\phi} \quad 1 \quad \operatorname{tr}\boldsymbol{M}\right. \tag*{52)}$$

where $\alpha$, $c$, and $\phi$ depend upon $\nu_0$, $\nu$, and $\operatorname{tr}\boldsymbol{D}$, and $\boldsymbol{M}$ is given as $\boldsymbol{M} = \nabla\nu\otimes\nabla\nu$. The coefficient $c$ is related to the cohesive properties of the material and $\phi$ is the internal angle of friction.

Though Eq. (51) is required to agree with the Mohr-Coulomb (or limiting equilibrium) criterion as $\boldsymbol{D}\to\boldsymbol{0}$, it is basically considered that this equilibrium limit is essentially different from that when flow occurs, however slowly. Savage [20] showed that in a rough-walled channel with an angle of inclination slightly less than the angle of repose, granular materials of uniform depth stay motionless in the absence of an external disturbance. However, giving a slight push the material is made to flow slowly but continuously at constant depth. It is, therefore evident that two different flow states exist at the same angle of inclination (but in general at different values of $\nu$.) In both cases the flow is assumed to be slow enough so that the inertia forces are assumed negligible and the equilibrium condition is achieved when the shear stresses balance the components of the body force in the direction of the flow. In even the simplest experiment in which the material is pushed, the shear stresses will result from different mechanisms. In the static case, they are due to dry interparticle friction and particle interlocking; whereas, in the shear-flow case, granules override other granules and the momentum transfer associated with interparticle collisions becomes more important. There does not seem to be a smooth transition from one state to another as $\boldsymbol{D} \rightarrow \boldsymbol{0}$, and thus it does not seem possible to describe both states by a single constitutive equation. Therefore, we shall regard $\boldsymbol{T}_e$ given by Eq. (51), as an additional component of stress arising during the deformation of the granular materials due to nonuniformity of $\nu$.

What we would like to do in this section is to show that Eq. (49) similar to Eq. (51), in the works of Cowin and Savage, will also predict the Mohr-Coulomb criterion. If the material is to comply with the Mohr-Coulomb criterion, it is necessary that the equilibrium normal stress and equilibrium shear stress acting on a particular plane at a particular point be related to each other through Eq. (3). Whereas in Newtonian fluids in equilibrium the shear stress always vanishes, in granular materials at equilibrium the shear stress has a specific nonzero value, which is related to the magnitude of the normal stress through Eq. (3). To obtain this relationship, we consider an arbitrary point on an arbitrary plane with normal $\boldsymbol{n}$.

Now, the normal stress $T$ acting on this plane is given by

$$T = (\boldsymbol{T}_e\boldsymbol{n}) \cdot \boldsymbol{n} \quad \text{or} \quad T = (T_e)_{ij} n_i n_j \,. \tag{53}$$

The magnitude of the stress vector $\boldsymbol{t}$ at this point is given by

$$|\boldsymbol{t}|^2 = \boldsymbol{t} \cdot \boldsymbol{t} = (\boldsymbol{T}_e\boldsymbol{n}) \cdot (\boldsymbol{T}_e\boldsymbol{n}) \,, \tag{54}$$

which is related to the shear and normal stresses, $S$ and $T$, through

$$(\boldsymbol{T}_e\boldsymbol{n}) \cdot (\boldsymbol{T}_e\boldsymbol{n}) = T^2 + S^2 \,. \tag{55}$$

That is, the square of the magnitude of the stress vector $\boldsymbol{t}$ is equal to the sum of the square of the normal stress $T$ and the square of the shear stress $S$ acting on the plane whose normal is $\boldsymbol{n}$. Let us see what type of restrictions will be imposed on Eq. (49).

The stress vector $\boldsymbol{t}$ on a surface whose normal is $\boldsymbol{n}$ is given by

$$\begin{aligned} \boldsymbol{t} = \boldsymbol{T}_e\boldsymbol{n} &= \{[\beta_0(\nu) + \beta_1(\nu)\, \nabla\nu \cdot \nabla\nu]\, \mathbf{1} + \beta_4(\nu, \nabla\nu)\, \nabla\nu \otimes \nabla\nu\}\, \boldsymbol{n} \\ &= [\beta_0(\nu) + \beta_1(\nu)\, |\nabla\nu|^2]\, \boldsymbol{n} + \beta_4(\nu, \nabla\nu)\, \frac{\partial\nu}{\partial n}\, \nabla\nu \,. \end{aligned} \tag{56}$$

Then the normal stress becomes

$$\begin{aligned} T = \boldsymbol{t} \cdot \boldsymbol{n} &= [\beta_0(\nu) + \beta_1(\nu)\, |\nabla\nu|^2]\, \boldsymbol{n} \cdot \boldsymbol{n} + \beta_4(\nu, \nabla\nu)\, \frac{\partial\nu}{\partial n}\, \nabla\nu \cdot \boldsymbol{n} \\ &= \beta_0(\nu) + \beta_1(\nu)\, |\nabla\nu|^2] + \beta_4(\nu, \nabla\nu) \left(\frac{\partial\nu}{\partial n}\right)^2 . \end{aligned} \tag{57}$$

And substituting Eq. (56) into Eq. (55) we have

$$\begin{aligned} T^2 + S^2 = {} & [\beta_0(\nu) + \beta_1(\nu)\, |\nabla\nu|^2]^2 + 2\beta_4(\nu, \nabla\nu)\, [\beta_0(\nu) \\ & + \beta_1(\nu)\, |\nabla\nu|^2] \left(\frac{\partial\nu}{\partial n}\right)^2 + {\beta_4}^2(\nu, \nabla\nu) \left(\frac{\partial\nu}{\partial n}\right)^2 |\nabla\nu|^2 \,. \end{aligned} \tag{58}$$

Now, let

$$
\begin{aligned}
s &= -\frac{\beta_4}{2}\,|\nabla \nu|^2\,, \\
t &= \beta_0(\nu) + \beta_1(\nu)\,|\nabla \nu|^2 + \frac{\beta_4}{2}\,|\nabla \nu|^2\,,
\end{aligned} \tag{59}
$$

then, Eq. (58) can be re-written as

$$
S^2 + (T - t)^2 = s^2\,,
$$

where if we consider $S$ and $T$ as Cartesian coordinates, we can see that Eq. (60) represents the equation for a circle centered at $S = 0$ and $T = t$, with radius $s$.

So far, the only thing we have said about $\beta_0$, $\beta_1$, and $\beta_4$ is that in 'equilibrium', they can depend on $\nu$. How these material properties are related to the actual physical parameters is another issue. Cowin [2], [45] and Savage [20] have provided some physical insight about the nature of $\beta_0$ and $\beta_1$. That is, since the material, prescribed with its constitutive relation $\boldsymbol{T}_e$ is to comply with the Mohr-Coulomb Criterion, then somehow $\beta_0$ amd $\beta_1$, and possibly $\beta_4$ should be related to cohesion and internal angle of friction. Thus, if we let

$$
\begin{aligned}
\beta_0 &= c\cot\phi\,, \\
\beta_1 &= \frac{\beta_4}{2}\left(\frac{1}{\sin\phi} - 1\right)
\end{aligned} \tag{61}
$$

and eliminate the term $\beta_4/2\,|\nabla \nu|^2$, in Eq. (59), between $s$ and $t$, we have

$$
s = \sin\phi(c\cot\phi - t)\,.
$$

Recall

$$
\begin{aligned}
s &= |S|/\cos\phi\,, \\
t &= T - |S|\tan\phi\,,
\end{aligned}
$$

then Eq. (62) can be re-written as

$$
|S| = c - T\tan\phi \tag{64}
$$

which is the same as Eq. (3), except for the negative sign which stems from the fact that we have assumed tensile stresses are positive. Therefore, we can see that with the interpretation of $\beta_0$ and $\beta_1$ given by Eq. (61), the constitutive relation proposed in this paper, Eq. (19), is also complying with the Mohr-Coulomb criterion.

## 3 Summary

In this paper we have tried to present the derivation of a constitutive relation for the stress tensor for granular materials, based on standard arguments in continuum mechanics. The equation has at least five material parameters which are undetermined. Due to the nonlinear terms the model is capable of predicting normal stress differences, which within the context of granular materials becomes the "dilatancy" phenomenon, known since Reynolds [59] and Reinèr [18]. At the same time, we have proposed a quadratic dependence of these material parameters on volume fraction. (An alternative way is to take a kinetic theory approach and try to find the exact dependence of these material parameters on primary variables. This study was taken up for this particular model by Boyle and Massoudi [79] and will not be discussed here). Appropriate restrictions such as a "limiting principle" and "densification in the lower regimes of the flow," provide further restrictions on these material parameters. In addition to these, if we require that the material is to comply with the Mohr-Coulomb criterion, then some of these material parameters would have to be given a different interpretation in such a way that they are now related to the internal angle of friction and the "cohesiveness" of the material.

In certain industrial processes such as flows in hoppers and bins [80], and fluidization, the solid particles are initially in static equilibrium. Then the flow slowly starts, due to action of gravity as is the case for bins and chutes, or due to the upward flow of a fluid in a fluidized bed. In the case of fluidization [81], the bed slowly expands. At this stage the yielding begins and the particles are no longer in static equilibrium. Frictional and sliding forces are the main deterrents to the flow. As the bed becomes fully fluidized, the particles begin to collide with each other and they move about rapidly. At this stage, the viscous and the interaction forces are the dominant mechanisms for flow. It is difficult to come up with a single constitutive relation which can cover the whole field of operation. In reality, in the regime where particles are colliding with each other, the fluid phase plays an important role, and thus the present model should only be used within the context of a multiphase mixture theory [82], [83].

Before we proceed to make a few comments about this and other similar models, for brevity we present the results here. The basic equation for the stress tensor is Eq. (19):

$$\begin{aligned} \boldsymbol{T} = {} & [\beta_0(\nu) + \beta_1(\nu)\,\nabla\nu\cdot\nabla\nu + \beta_2(\nu)\,\mathrm{tr}\,\boldsymbol{D}]\,\mathbf{1} \\ & + \beta_3(\nu)\,\boldsymbol{D} + \beta_4(\nu,\nabla\nu)\,\nabla\nu\otimes\nabla\nu + \beta_5(\nu)\,\boldsymbol{D}^2\,, \end{aligned} \tag{19}$$

where if the material is "fully" flowing, the following representations are proposed for the $\beta$'s:

$$\beta_0(\nu) = \varkappa_1\nu\,, \tag{35}$$

$$\beta_1(\nu) = \beta_{10} + \beta_{11}\nu + \beta_{12}\nu^2\,, \tag{37}$$

$$\beta_2(\nu) = \beta_{20} + \beta_{21}\nu + \beta_{22}\nu^2\,, \tag{38}$$

$$\beta_3(\nu) = \beta_{30} + \beta_{31}\nu + \beta_{32}\nu^2\,, \tag{33}$$

$$\beta_4(\nu) = \beta_{40} + \beta_{41}\nu + \beta_{42}\nu^2\,, \tag{39}$$

$$\beta_5(\nu) = \beta_{51} + \beta_{51}\nu + \beta_{52}\nu^2\,, \tag{34}$$

where

$$\beta_{30} = \beta_{20} = \beta_{50} \tag{40}$$

due to the "limiting principle." In their studies, Rajagopal et al. [84] proved existence of solutions, for a selected range of parameters, when,

$$\begin{aligned} & \beta_1 + \beta_4 > 0 \\ & \varkappa_1 < 0\,. \end{aligned} \tag{65}$$

For other rheological parameters, i.e., $\beta_2$, $\beta_3$, and $\beta_5$, we can use the experience in the mechanics of non-Newtonian fluids to find out more information about the signs. Obviously, since $\beta_3$ is related to the shear viscosity, we assume it is positive. Perhaps a thermodynamic or a stability analysis would reveal further information about the signs and the relative importance of these parameters.

If the material is just about to yield, then if we are to comply with the Mohr-Coulomb criterion, the following representations are to be given to the material parameters in Eq. (19):

$$\begin{aligned}\beta_0 &= c\cot\phi\,,\\ \beta_1 &= \frac{\beta_4}{2}\left(\frac{1}{\sin\phi}-1\right),\end{aligned}\tag{61}$$

where $\phi$ is the internal angle of friction and $c$ is a coefficient measuring cohesion. The interpretation for $\beta_2$, $\beta_3$, and $\beta_5$ would stand as the previous case, if we accept that the stress tensor can be decomposed into an "equilibrium" part and a "dynamic" part, as discussed earlier, where $\beta_2$, $\beta_3$, and $\beta_5$ are responsible for the dynamic effects.

In conclusion, we would like to mention that the model presented by Goodman and Cowin [40], [41] required "higher" balance laws. As such, that model, in a sense, is more similar to the liquid crystal models [85], [86], [87], [88] and the micropolar theories [89]. The present model, though similar in structure to theirs, does not require any further balance laws. At the same time, the equilibrium part of the stress looks very similar to a model proposed by Dunn and Serrin [90] where the effects of interstitial working was considered for a class of fluids, similar to the Korteweg-type fluids. Again, the interpretation of the material parameters, in some cases is different from the ones given by Dunn and Serrin. For example, the coefficient similar to our $\beta_4$ in their model is called the surface tension coefficient, whereas in the present model, $\beta_4$ is interpreted as a parameter responsible for the re-arrangement (due to density gradients) of the particles, and it is a primary parameter for the nonlinear effect of dilatancy. Also, if $\beta_1 = \beta_4 = 0$, that is, if the density (or volume fraction) gradients are ignored, Eq. (19) reduces to the Reiner-Rivlin fluid model [18], [72], provided that we interpret $\beta_0$ as the pressure, $\beta_3$ as the coefficient of the viscosity, and $\beta_5$ as the coefficient of the cross-viscosity.

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